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TWO CONCEPTS OF POSITIVE DEPENDENCE, WITH APPLICATIONS IN MULTI--ETC(U)

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

2

1. REPORT NUMBER: 18 19 AFOSR-TR- 79-0429

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle): 6 TWO CONCEPTS OF POSITIVE DEPENDENCE, WITH APPLICATIONS IN MULTIVARIATE ANALYSIS

5. TYPE OF REPORT & PERIOD COVERED: 9 Interim rept.,

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s): 10 Abdul-Hadi N. Ahmed, Naftali A. Langberg, Ramon V. Leon, ~~and~~ Frank Proschan

8. CONTRACT OR GRANT NUMBER(s): 15 AFOSR-78-3678

9. PERFORMING ORGANIZATION NAME AND ADDRESS: The Florida State University, Department of Statistics, ~~Tampa~~ TALAHASEE, FL 32306

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 16 2304/A5 17 A5

11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB, Washington, DC 20332

12. REPORT DATE: 1979

13. NUMBER OF PAGES: 39

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): 11 Nov 78

LEVEL II

15. SECURITY CLASS. (of this report): UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report): 12 39 p.

DDC APR 18 1979 C

18. SUPPLEMENTARY NOTES: 14 FSU-STATISTICS-M486, TR-78-6

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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AFOSR-TR- 79-0429

Two Concepts of Positive Dependence, with Applications in Multivariate Analysis

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Statistics Report M486
AFOSR Technical Report No. 78-6

November, 1978

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[1]Research supported by the Air Force Office of Scientific Research under AFOSR Grant 78-3678. The United States Government is authorized to reproduce and distribute reprints for governmental purposes.

AMS Classification Numbers: 62H99 & 62N05

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ABSTRACT

We develop properties and theory for positive orthant dependence, a multivariate extension of Lehmann's positive quadrant dependence, and right tail increasing in sequence dependence, a multivariate extension of Esary and Proschan's bivariate right tail increasing dependence. Applications are then obtained in the form of inequalities and monotonicity in a wide variety of multivariate statistical problems, including MANOVA, contingency tables, dependence measurement, competing risk models, reliability of series systems, and distribution theory.

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1. Introduction and Summary.

A variety of qualitative concepts of positive dependence have been defined, studied, and applied in reliability, in areas of statistics such as analysis of variance, multivariate tests of hypotheses, sequential testing, and in probability inequality theory. (See, e.g., Esary, Proschan and Walkup (1967), Šidák (1958), Mallow (1968), Brindley and Thompson (1972), Yanagimoto (1972), Serfling (1975), Barlow and Proschan (1975), Alam and Wallenius (1976), Kemperman (1977), Tong (1977a, 1977b), Shaked (1977), Jogdeo (1977, 1978), and Dykstra and Hewett (1978), among others).

In the present paper we study multivariate versions of bivariate positive dependence, namely positive quadrant dependence (introduced by Lehmann, 1966) and right tail increasing dependence (introduced by Esary and Proschan, 1972). In Section 3 we develop the basic theory for positively orthant dependent (POD) random vectors (see Def. 2.4); POD is the multivariate version of positive quadrant dependence. We point out that POD random vectors enjoy properties analogous to the basic properties possessed by associated random variables. (See Esary, Proschan, and Walkup, 1967.) In addition, we obtain several basic preservation properties for POD random vectors. These permit us to extend the class of POD random vectors to include a large number of cases of practical interest.

In Section 4 we develop the basic theory of right tail increasing in sequence (RTIS) random vectors; see Def. 4.1. We point out that the RTIS property, although similar in concept to the earlier conditionally increasing in sequence (CIS) property (see Def. 2.2), neither implies nor is implied by the

CIS property. However, both RTIS and CIS yield the useful POD inequality (2.1). We develop several easily checked conditions for demonstrating that a random vector is RTIS. Finally, we show that RTIS random vectors arise in certain cases of sampling when the underlying distribution possesses a random parameter.

In Section 5 we present a sampling of useful applications of the theory developed in Sections 3 and 4 for POD and RTIS random vectors, respectively. These include:

(a) The MANOVA problem with known covariance matrix,

(b) A characterization of independence in 2 x 2 contingency tables,

(c) The demonstration of RTIS for random vectors governed by certain well known distributions or sampling schemes,

(d) Many well known measures of association are shown to be positive in commonly occurring sampling situations; these include Kendall's τ, Spearman's ρ_s, and Blomqvist's q.

(e) Competing risk model with proportional failure rates, mutual independence not assumed; the model is applicable in both biometry and reliability,

(f) Probability inequalities using POD, applicable to the absolute normal distribution with random means, the multivariate noncentral t distribution, and the multivariate gamma distribution. Additional applications exist, but are left for subsequent papers.

2. Preliminaries.

In this section we present definitions, notations, and basic facts used throughout the paper.

We use "increasing" in place of "nondecreasing" and "decreasing" in place of "nonincreasing" throughout.

2.1. Definition (Karlin, 1968). A function $f: R^2 \to [0, \infty)$ is totally positive of order 2 (TP_2) if

$$\begin{vmatrix} f(x_1, y_1) & f(x_1, y_2) \\ f(x_2, y_1) & f(x_2, y_2) \end{vmatrix} \geq 0$$

for each choice $x_1 < x_2$, $y_1 < y_2$.

2.2. Definition. The random variables $X_1, \ldots, X_n$ are conditionally increasing in sequence (CIS) if for $i = 2, 3, \ldots, n$, and all real x_i,

$$P[X_i > x_i | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}]$$

is increasing in $x_1, \ldots, x_{i-1}$.

2.3. Definition (Esary, Proschan; and Walkup, 1967). The random variables $X_1, \ldots, X_n$ are associated if

$$\mathrm{Cov}[f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)] \geq 0$$

for all increasing real valued Borel measurable functions f, g for which the covariance exists.

2.4. Definition. The random variables $X_1, \ldots, X_n$ are mutually **positively orthant dependent** (POD) if

$$(2.1) \qquad P\left[\bigcap_{i=1}^{n} (X_i > x_i)\right] \geq \prod_{i=1}^{n} P(X_i > x_i)$$

for all real numbers $x_1, \ldots, x_n$. For $n = 2$, POD coincides with positive quadrant dependence (PQD) (Lehmann, 1966).

2.5. Definition (Barlow and Proschan, 1975). Let n be an integer exceeding 2. A function $f: R^n \to [0, \infty)$ is said to be **totally positive of order 2 in pairs** (TP_2 in pairs) if for any pair of arguments x_i, x_j, $f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$, viewed as a function of x_i, x_j with the remaining arguments fixed, is TP_2.

2.6. Remark. Def. 2.4 differs from the one used in the literature (see, for example, Dykstra, Hewett, and Thompson, 1973) in that we use the right tail (survival) probabilities for our definition and they use the left tail probabilities. More precisely, Dysktra et al. (1973) define $X_1, \ldots, X_n$ to be POD if

$$(2.2) \qquad P\left[\bigcap_{i=1}^{n} (X_i \leq x_i)\right] \geq \prod_{i=1}^{n} P(X_i \leq x_i)$$

The reader should notice that (2.1) and (2.2) coincide only if $n = 2$. However, for $n > 2$, the two expressions are not equivalent, for if we let X_1, X_2, and X_3 be random variables taking values (2, 1, 1), (1, 2, 1), (1, 1, 2), and (2, 2, 2) each with probability 1/4, then simple calculations yield

$$P[\bigcap_{i=1}^{3} (X_i > 1)] = 1/4 > 1/8 = \prod_{i=1}^{3} P(X_i > 1),$$

while

$$P[\bigcap_{i=1}^{3} (X_i \leq 1)] = 0 < 1/8 = \prod_{i=1}^{3} P(X_i \leq 1).$$

As a matter of fact, POD is introduced as a weaker concept of positive dependency among random variables than association. Since association implies both (2.1) and (2.2) (Barlow and Broschan, 1975, Theorem, 1975, Theorem 3.2, p. 33), there is no reason to prefer one definition over the other, except that Definition 2.4 is more directly meaningful in the reliability context.

<u>2.7. Lemma.</u> $f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$ is TP_2 in pairs $\Rightarrow X_1, \ldots, X_n$ are CIS $\Rightarrow X_1, \ldots, X_n$ are associated $\Rightarrow X_1, \ldots, X_n$ are associated $\Rightarrow$ $X_1, \ldots, X_n$ are POD.

(A proof of Lemma 2.7 appears in Esary, Proschan, and Walkup, 1967.)

3. Positively Orthant Dependent Random Variables.

In this section we present theoretical results about POD random variables. Before introducing the main results of this section, let us present some of the desirable properties of POD random variables.

It is fairly easy to prove that

(P_0) Any set of independent random variables is POD.

(P_1) Any subset of POD random variables is POD.

(P_2) The set consisting of a single random variable is POD.

(P_3) If $X_1, \ldots, X_n$ are POD and $g_1, \ldots, g_n$ are real valued increasing functions, then $g_1(X_1), \ldots, g_n(X_n)$ are POD.

(P_4) The union of independent sets of POD random variables are POD.

The multivariate exponential (Marshall and Olkin, 1967), the multivariate absolute value normal (Šidak, 1973), the negative multinomial, and the multivariate negative hypergeometric (Jogdeo and Patil, 1975) are typical examples of distributions of POD random variables. For other interesting examples, see Ahmed, León, and Proschan (1978).

Finally, it may be of interest to note that pairwise POD does not imply mutually POD. To see this, consider the following example.

Example. Let Ω be the set of equally likely integers, w, where $1 \le w \le 8$. Then let $A_1 = \{1, 2, 3, 4\}$, $A_2 = \{1, 2, 5, 6\}$, and $A_3 = \{3, 4, 5, 6\}$ be subsets of Ω. Further, let X_i be the random variable defined by the indicator function of the set A_i for $i = 1, 2$, and 3.

Clearly, (X_i, X_j) are independent, hence POD, for all $i \neq j$; $i, j = 1, 2$, and 3. Therefore, X_1, X_2, and X_3 are pairwise POD.

However, $P[\bigcap_{i=1}^{3} (X_i > 0)] = P(\bigcap_{i=1}^{3} A_i) = P(\phi) = 0$,

where ϕ denotes the empty set,

and

$$P(X_i > 0) = 1/2 \text{ for all } i = 1, 2, \text{ and } 3.$$

It follows that

$$P[\bigcap_{i=1}^{3} (X_i > 0)] = 0 < 1/8 = \prod_{i=1}^{3} P(X_i > 0).$$

Thus X_1, X_2, and X_3 are not mutually POD. Next, we show that POD is "inherited" under certain commonly encountered mixing processes.

The proofs of the following two subsections will make considerable use of some of the properties of positive quadrant dependence and association established in Esary, Proschan, and Walkup (1967), and Esary and Proschan (1972).

For convenience and ease of reference, the necessary definitions, terminology, and basic facts are listed below.

3.0.1. Definition. A random vector $\underline{Y}$ is said to be stochastically increasing in the random vector $\underline{X}$ if

$$E[f(\underline{Y})|\underline{X} = \underline{x}]$$

is increasing in $\underline{x}$ for every increasing real valued integrable function f; we write $\underline{Y}\uparrow$st. in $\underline{X}$.

3.0.2. Theorem. X_1 and X_2 are positively quadrant dependent (PQD) if and only if

$$\mathrm{Cov}[f(X_1), g(X_2)] \geq 0$$

for all increasing real valued Borel measurable functions f, g for which the covariance exists.

The following properties hold for associated random variables:

(P_5) The set consisting of a single random variable is associated.

(P_6) Increasing functions of associated random variables are associated.

We are now in a position to present the principal results of this section.

3.1. PQD Random Variables.

Although the theoretical results of this section and the section to follow may be stated for probability distributions defined on a general measure space with a partial ordering, for our applications it suffices to consider probability distributions defined on a measurable rectangle in R^n (n-dimensional Euclidean space) endowed with the usual componentwise partial ordering.

3.1.1. Theorem. Let (a) X_1, X_2, given $\underline{Y}$, be conditionally PQD, (b) $X_i \uparrow$st. in $\underline{Y}$ for i = 1, 2, and (c) $\underline{Y}$ be associated. Then X_1, X_2 are PQD.

The same conclusion is true if in (b) $\downarrow$ replaces $\uparrow$.

Proof. Let f, g be increasing real valued Borel measurable bounded functions. In view of Theorem 3.0.2, it is enough to show that

$$\mathrm{Cov}[f(X_1), g(X_2)] \geq 0.$$

Note that

$$(3.1) \qquad \mathrm{Cov}[f(X_1), g(X_2)] = E\{\mathrm{Cov}[f(X_1), g(X_2)]|\underline{Y}\} + \mathrm{Cov}\{E[f(X_1)|\underline{Y}], E[g(X_2)|\underline{Y}]\}.$$

Conditioned on $\underline{Y}$, X_1, X_2 are PQD. Thus, by Th. 3.0.2, the first term on the right side of (3.1) is nonnegative.

From Def. 3.0.1, the conditional expectations in the second term on the right of (3.1) are increasing functions of $\underline{Y}$. By assumption, $\underline{Y}$ is associated. Thus by (P_6), the covariance of the conditional expectations in the second term is nonnegative. It follows that

$$\mathrm{Cov}[f(X_1), g(X_2)] \geq 0.$$

Thus, X_1, X_2, are PQD. ||

We now turn to a generalization of the above result for:

3.2. POD Random Variables.

Let $\underline{X} = (X_1, \ldots, X_n)$ and $\underline{Y} = (Y_1, \ldots, Y_m)$. We shall find the following theorem useful in applications.

3.2.1. Theorem. Let (a) $\underline{X}$, given $\underline{Y}$, be conditionally POD, (b) $X_i \uparrow$st. in $\underline{Y}$ for i = 1, 2, ..., n, and (c) $\underline{Y}$ be associated. Then (1) ($\underline{X}$, $\underline{Y}$) are POD, and (2) in particular, $\underline{X}$ is POD. The same conclusion hold if in (b) $\downarrow$ replaces $\uparrow$.

Unfortunately, the elementary proof of Theorem 3.1.1. for the bivariate case(PQD) does not extend to higher dimensions (POD). For this reason we present the following lemma, which is of interest in its own right, since it yields a theorem of Dykstra, Hewett, and Thompson (1973, Sec. 3, Theorem 1) and constitutes a generalization of Kimball's (1951) result. The lemma will play a crucial role in proving Theorem 3.2.1.

3.2.2. Lemma. If $Y_1, \ldots, Y_m$ are associated and if $g_i(y_1, \ldots, y_m)$

are nonnegative and increasing for $i = 1, 2, \ldots, k$, $k \geq 2$; then

$$(3.2) \qquad E\left[\prod_{i=1}^{k} g_i(Y_1, \ldots, Y_m)\right] \geq \prod_{i=1}^{k} E[g_i(Y_1, \ldots, Y_m)].$$

The same inequality is true if the g's are decreasing instead of increasing. If $k = 2$, and the expectations exist, we may omit the requirement that the g_i are nonnegative.

Proof. We shall prove the lemma by induction.

Suppose $k = 2$. By the definition of association (Def. 2.3), and the fact that $g_i(y_1, \ldots, y_m)$ is increasing in all arguments, the inequality immediately follows from (P_6).

Now suppose (3.2) holds for $k - 1$; i.e.,

$$(3.3) \qquad E\left[\prod_{i=1}^{k-1} g_i(Y_1, \ldots, Y_m)\right] \geq \prod_{i=1}^{k-1} E[g_i(Y_1, \ldots, Y_m)].$$

Again by (P_6); $\prod_{i=1}^{k-1} g_i(Y_1, \ldots, Y_m)$ and $g_k(Y_1, \ldots, Y_m$ are associated.

It follows that

$$(3.4) \qquad E\left[\prod_{i=1}^{k-1} g_i(Y_1, \ldots, Y_m) \geq Eg_k(Y_1, \ldots, Y_k) \cdot E\left[\prod_{i=1}^{k-1} g_i(Y_1, \ldots, Y_m)\right].$$

Combining (3.3) and (3.4), we obtain the conclusion of the lemma. ||

3.2.3. Remark. The conclusion of Lemma 3.2.2. was obtained by Dyskra, Hewett, and Thompson (1973) under the assumption that $Y_1, \ldots, Y_m$ are CIS, which is a stronger assumption than our assumption of association in Lemma 3.2.2. Furthermore, if $m = 1$, Lemma 3.2.2 can be used to generalize Kimball's (1951) theorem since any real random variable is associated.

<u>Proof of Theorem 3.2.1.</u>

(1) Observe that

$$P[\bigcap_{i=1}^{n} (X_i > x_i), \bigcap_{j=1}^{m} (Y_j > y_j)] = E_{\underline{Y}}\{P[\bigcap_{i=1}^{n} (X_i > x_i) | \underline{Y}] \cdot I_{\{\bigcap_{j=1}^{m} (Y_j > y_j)\}}\}$$

$$\geq E_{\underline{Y}}\{\prod_{i=1}^{n} P[X_i > x_i | \underline{Y}] \cdot I_{\{\bigcap_{j=1}^{m} (Y_j > y_j)\}}\}$$

$$\geq \prod_{i=1}^{n} E_{\underline{Y}} P(X_i > x_i | \underline{Y}) \cdot E_{\underline{Y}} I_{\{\bigcap_{j=1}^{m} (Y_j > y_j)\}}$$

$$= \prod_{i=1}^{n} P(X_i > x_i) \cdot E_{\underline{Y}} I_{\{\bigcap_{j=1}^{m} (Y_j > y_j)\}}$$

$$\geq \prod_{i=1}^{n} P(X_i > x_i) \cdot \prod_{j=1}^{m} P(Y_j > y_j).$$

The first inequality follows from assumption (a). The second and the third inequalities follow from assumptions (b), (c), together with Lemma 2.3.2. Thus $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ are POD.

(2) The result follows immediately from (P_1). ||

<u>3.2.4. Corollary.</u> Let (a) $\underline{X}$ given Z, a scalar random variable, be conditionally POD, and (b) $X_i \uparrow$ st. in Z for i = 1, ..., n. Then $\underline{X}$ are POD.

<u>Proof.</u> The proof is an immediate consequence of Th. 3.2.1 and (P_5). ||

Cor. 3.2.4 is of particular interest for obtaining applications of value

in probability and statistical theory. In statistics mixtures of distributions arise in a variety of circumstances. In the area of statistical decision theory the varizble Z plays the role of a parameter having a prior distribution. For most problems in multiple comparisons, usually the probability of a correct decision can be given in a form (see, for example, Tong,1977b) defined by a mixture of distributions.

3.2.5. Remark. The conclusion of Cor. 3.2.4 may not hold if the assumption $X_i \uparrow$ st. in Z for $i = 1, \ldots, n$ is dropped, as shown by the following:

3.2.6. Example. For Z = 1, 2, let

$\{X_1^{(1)} = 0$ a.s., $X_2^{(1)}$ be uniformly distributed on $[0, 1]\}$ with probability 1/2,

and

$\{X_1^{(2)}$ be uniformly distributed on $[0, 1]$, $X_2^{(2)} = 0$ a.s$\}$ with probability 1/2.

Clearly, X_1, X_2 given Z, are conditionally independent, so that they are conditionally POD.

However, $EX_1X_2 = 0$ and $EX_1 = EX_2 = 1/4$. Thus $Cov(X_1, X_2) < 0$. Hence X_1, X_2 are not POD. ||

Next, we show that the property of POD among random variables can be created and preserved through suitable combinations and transformations of random variables.

3.3. Preservation Properties of POD.

The following statements are true. The proof in each case follows from Th. 3.1.1, (P_3), and the fact that the pair (X, X) is PQD.

(i) Let $Y = g(X) + Z$, where Z is independent of X and g: $R \to R$ is an increasing Borel measurable function. Then (X, Y) is PQD.

(ii) For $ab > 0$, (U, V) is PQD, and Z be independent of (U, V), define:

$$X = U + aZ \quad \text{and} \quad Y = V + bZ .$$

Then (X, Y) is PQD.

(iii) Let (U, V) be PQD, Z be independent of (U, V), and f, g: $R^2 \to R$ be Borel measurable functions. Define:

$$X = f(U, Z), \quad Y = g(V, Z).$$

Let f, g be increasing in Z but otherwise arbitrary. Then (X, Y) is PQD.

(iv) Let $\underline{X} = (X_1, \ldots, X_n)$ be POD, g_i: $R \to R$ be Borel measurable increasing functions for $i = 1, 2, \ldots, n$. Let Z be independent of $\underline{X}$. Define $Y_i = g_i(X_i) + Z$, $i = 1, \ldots, n$. Then $Y_1, \ldots, Y_n$ are POD.

4. An Extension of a Concept of Positive Dependence.

In this section we extend a useful concept of bivariate positive dependence to its multivariate version. We develop sufficient conditions for this type of dependence, and derive some inequalities and monotonicity of conditional survival functions that result from it. The inequalities can be used to determine whether over-(or under-) estimates occur when one acts as if positively dependent random variables are independent.

Esary and Proschan (1972) define a random variable Y to be right tail increasing (RTI) in a random variable X if

$$P[Y > y | X > x]$$

is increasing in x for all real numbers y.

In the multivariate case it is somewhat surprising that no analog of this concept has yet been proposed. In this section we define a natural multivariate version of RTI, namely, right tail increasing in sequence (RTIS). We give sufficient conditions for its existence, and show how it relates to other known multivariate dependence notions.

4.1. Definition. A sequence of random variables $X_1, \ldots, X_n$ is said to be right tail increasing in sequence (RTIS) if

$$P[X_{i+1} > x_{i+1} | \bigcap_{j=1}^{i} (X_j > x_j)]$$

is increasing in $x_1, \ldots, x_i$, for $i = 1, 2, \ldots, n - 1$.

4.2. Remark. It is easy to verify that RTIS $\Rightarrow$ POD. Thus, showing that a random vector is RTIS immediately yields Inequality (2.1).

Since $X_2 \uparrow$ st. in $X_1 \Rightarrow X_2$ RTI in X_1 (See, Esary, and Proschan, 1972), it may be tempting to conjecture that $X_1, \ldots, X_n$ *CIS* $\Rightarrow X_1, \ldots, X_n$ *RTIS*. However, this is not always true. To see this, consider the following.

4.3. Example. Let X_3, given $(X_1, X_2) = (x_1, x_2)$ be distributed according to the normal $N(x_1 + x_2, 1)$, and let (X_1, X_2) be jointly distributed according to

$X_1 \backslash X_2$	a	b	c
a	.1	.2	.0
b	.1	.0	.2
c	.0	.1	.3

where a < b < c.

Clearly, $X_3 \uparrow$ st. in (X_1, X_2) and $X_2 \uparrow$ st. in X_1. Thus, X_1, X_2, and X_3 are CIS. However, it is easy to check that

$$P[X_3 > a | X_1 > b, X_2 > a] < P[X_3 > a | X_1 > a, X_2 > a];$$

i.e., X_1, X_2, and X_3 are not RTIS.

A similar example can easily be constructed to show that RTIS $\not\Rightarrow$ CIS. Thus neither concept implies the other.

Motivated by Example 4.2, we present easily checked sufficient conditions for a sequence of random variables to be RTIS.

The following theorem can be applied (see Subsection 5.3) in a very general context to deduce monotonicity of the contional survival probabilities and to obtain general inequalities for a wide variety of standard multivariate distributions and processes.

4.4. Theorem. Let $\bar{F}_n(x_1, \ldots, x_n) \equiv P[\bigcap_{i=1}^{n}(X_i > x_i)]$ be TP_2 in each pair of arguments for fixed values of the remaining arguments. Then $X_1, \ldots, X_n$ are RTIS. Moreover, every permutation of $X_1, \ldots, X_n$ is RTIS.

Proof. Let $x_3, \ldots, x_n$ be fixed at $-\infty$. Then $\bar{F}_2(x_1, x_2)$ is TP_2 in $-\infty < x_1, x_2 < \infty$. By a result of Barlow and Proschan (1975, Th. 4.2, p. 143), X_2 is RTI in X_1.

For fixed x_2, $\bar{F}_3(x_1, x_2, x_3)$ is TP_2 in $-\infty < x_1, x_3 < \infty$. Thus

$P[X_3 > x_3 | X_2 > x_2, X_1 > x_1]$ is $\uparrow$ in x_1 for all x_3.

By symmetry, $P[X_3 > x_3 | X_2 > x_2, X_1 > x_1]$ is $\uparrow$ in x_2 for all x_3. It follows that

$P[X_3 > x_3 | X_2 > x_2, X_1 > x_1]$ is $\uparrow$ in x_1, x_2 for all choices of x_3, so that X_3 is RTI in (X_1, X_2). Repetition of this argument yields the desired result that X_i is RTI in $(X_1, \ldots, X_{i-1})$ for $i = 2, \ldots, n$. Thus $X_1, \ldots, X_n$ are RTIS. By symmetry, every permutation of $X_1, \ldots, X_n$ is RTIS. ||

4.5. Remark. It is true that the hypothesis of Th. 4.3, together with an additional condition (See Kemperman, 1977, Sec. 6, Assertions (i) and (iii)) also yield CIS which implies POD. However, it is important to note that the inequalities based on the RTIS concept are different in nature from those obtained using the CIS concept. Moreover, it is obvious from Definition 4.1 that the amount of information needed to demonstrate RTIS is much less than that needed to demonstrate CIS. In addition, there are situations, particularly in reliability theory and biological studies (see Subsections 5.3 and 5.5), in which RTIS and not CIS is the relevant concept.

Another interesting sufficient condition for $X_1, \ldots, X_n$ to be RTIS is contained in:

4.6. Theorem. Let $X_1, \ldots, X_n$, given λ, a scalar random variable, be conditionally i.i.d with common TP_2 density $f(x_i, \lambda)$. Then $X_1, \ldots, X_n$ are RTIS. Moreover, every permutation of $X_1, \ldots, X_n$ is RTIS.

Before we prove Th. 4.5., we present the forllowing notation and definition.

Let $\underline{x}^{(i)} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, i.e., the vector obtained from $\underline{x}$ by deleting x_i. We define $\underline{X}$ to be totally positive by deletion (TPD) if $\underline{X}$ has a density h satisfying

$$\begin{vmatrix} h(x_1, \ldots, x_n) & h(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n) \\ h(x_1', \ldots, x_{i-1}', x_i, x_{i+1}', \ldots, x_n') & h(x_1', \ldots, x_n') \end{vmatrix} \geq 0$$

for all $x_i \leq x_i'$, $i = 1, \ldots, n$. (This notion is called m^*-positive by Alam and Wallenius, 1976).

We will break the proof of Th. 4.5 into a sequence of lemmas as follows.

4.7. Lemma. Let (a) $X_1, \ldots, X_n$, given λ, a random variable, be independent random variables with a common TP_2 density $f(x_i, \lambda)$. Then $\underline{X}$ is TPD.

Proof. Since $f(x_i, \lambda)$ is TP_2, then

$$(4.2) \qquad \begin{vmatrix} f(x_i, \lambda) & f(x_i', \lambda) \\ f(x_i, \lambda') & f(x_i', \lambda') \end{vmatrix} \geq 0$$

whenever $x_i \leq x_i'$ and $\lambda \leq \lambda'$.

Multiplying the top row of (4.2) by $\prod_{j \neq i,\, i=1}^{n} f(x_j, \lambda)$ and the bottom row by $\prod_{j \neq i,\, i=1}^{n} f(x_j', \lambda')$, where $x_j \leq x_j'$ for all $j = 1, \ldots, n$, $j \neq i$, we obtain:

$$\begin{vmatrix} h(x_1, \ldots, x_i, \ldots, x_n; \lambda) & h(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n; \lambda) \\ h(x_1', \ldots, x_{i-1}', x_i, x_{i+1}', \ldots, x_n'; \lambda) & h(x_1', \ldots, x_n'; \lambda') \end{vmatrix} \geq 0$$

for every $x_i \leq x_i'$, $i = 1, \ldots, n$, and $\lambda \leq \lambda'$. Thus $\underline{X}$ is TPD. ||

Next we need the following:

4.8. Definition (Harris, 1970). A set of random variables $X_1, \ldots, X_n$ is said to be right corner set increasing (RCSI) if

$$P[\bigcap_{i=1}^{n} (X_i > x_i) \mid \bigcap_{i=1}^{n} (X_i > x_i')]$$

is increasing in $x_1', \ldots, x_n'$ for every choice of $x_1, \ldots, x_n$.

4.9. Lemma. If $X_1, \ldots, X_n$ are RCSI, then $X_1, \ldots, X_n$ are RTIS. Moreover, every permutation of $X_1, \ldots, X_n$ is RTIS.

Proof. $X_1, \ldots, X_n$ are RCSI. Thus

$$P[\bigcap_{i=1}^{n} (X_i > x_i) \mid \bigcap_{i=1}^{n} (X_i > x_i')]$$

is increasing in $x_1', \ldots, x_n'$ for all choices of $x_1, \ldots, x_n$. Therefore, for fixed j,

$$\frac{P[X_j > x_j, \bigcap_{i=1}^{n} (X_i > x_i')]}{P[\bigcap_{i=1}^{n} (X_i > x_i')]}$$

is increasing in $x_1', \ldots, x_n'$ for all choices of x_j.

Now letting $x_i' \to -\infty$ for all $i = 1, \ldots, j-1$, we obtain

$$P[X_j > x_j | X_{j-1} > x_{j-1}', \ldots, X_1 > x_1']$$

is increasing in $x_1', \ldots, x_{j-1}'$ for all choices of x_j. Since j is arbitrary, $X_1, \ldots, X_n$ are RTIS. By symmetry, every permutation of $X_1, \ldots, X_n$ is RTIS. ||

Proof of Theorem 4.6. By Lemma 4.7, $\underline{X}$ is TPD. By Cor. 3.4 of Alam and Wallenius (1976), $X_1, \ldots, X_n$ are RCSI. By Lemma 4.9, the proof follows. ||

For our next result we need the following definition.

4.10. Definition. A set of n distribution functions $F_1^{(\lambda)}, \ldots, F_n^{(\lambda)}$ is said to have conditional proportional hazard functions if

$$F_i^{(\lambda)}(x) = 1 - e^{-\lambda c_i R(x)}, \quad i = 1, \ldots, n,$$

where $c_1, \ldots, c_n$ are positive constants, λ is nonnegative, and $R(x)$ is an increasing function with $R(0-) = 0$ and $R(\infty) = \infty$.

When $R(x) = x^{\alpha}$, $\alpha > 0$, we obtain the familiar Weibull family which contains the exponential family for $\alpha = 1$.

The following theorem has interesting applications involving competing risks or fatigue models; see Subsection 5.5.

4.11. Theorem. Let $F_1^{(\lambda)}, \ldots, F_n^{(\lambda)}$ have conditional proportional hazard functions, where λ is distributed according to G. Then

(1) $\bar{F}_{\underline{X}}(x_1, \ldots, x_n) = \int \prod_{i=1}^{n} \bar{F}_{X_i}^{(\lambda)}(x_i)dG(\lambda)$ is TP_2 in pairs, and (2) $\underline{X}$ is RTIS.

Proof. (1) For fixed i, j, let x_k be fixed for all $k \neq i$, $k \neq j$, and $k = 1, 2, \ldots, n$. We may write

$$\bar{F}_{\underline{X}}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = \int \bar{F}_{X_i}^{(\lambda)}(x_i)\bar{F}_{X_j}^{(\lambda)}(x_j)\prod_{\substack{k=1 \\ k \neq i,j}}^{n} \bar{F}_{X_k}^{(\lambda)}(x_k)dG(\lambda).$$

Since $\bar{F}_{x_i}^{\lambda}$ is TP_2 in $0 < x_i, \lambda < \infty$, and $\bar{F}_{x_j}^{(\lambda)}$ is TP_2 in $0 < x_j, \lambda < \infty$, we use the composition theorem (Karlin, 1968, p. 17) to obtain the desired conclusion.

(2) The desired conclusion is an immediate consequence of Th. 4.6. ||

5. Applications To Statistics and Probability.

5.1. The MANOVA Problem With Known Covariance Matrix.

In this type of problem, one observes a normally distributed pxr random matrix $[X]$ with $E[X] = [\mu]$. The columns of the matrix are independent with common covariance Σ which can be taken, without any loss of generality, as $I_{p \times p}$.

The problem is to test the hypothesis:

$$H_0: \ [\mu] = [0]$$

vs.

$$H_1: \ [\mu] \neq [0]$$

The maximal invariant test statistic for this type of problem is

$$\ell \equiv (\ell_1, \ldots, \ell_t),$$

where $t = \min(p, r)$ and $\ell_1 \geq \ell_2 \geq \ldots \geq \ell_t > 0$ are the nonzero characteristic roots of XX'.

To study the distribution of $\underline{\ell}$, one can assume that $[\mu] = D_\lambda$ (a diagonal matrix with the λ's on its main diagonal), since ℓ is invariant. Then XX' has the same distribution as TT', where $T_{p \times p}$ is now a random lower triangular matrix whose elements $T_{ij} (i \geq j)$ are mutually independent with the following distributions:

$$T_{11} \sim [\chi_n^2(\lambda_1)]^{1/2},$$

$$T_{ii} \sim [\chi^2_{n-i+1}]^{1/2}, \ 2 \leq i < p,$$

$$T_{ij} \sim N(0, 1) \quad , \ i > j.$$

For $p = 2$, if rank $(\mu) \geq 2$, i.e., if $\lambda_1, \lambda_2 > 0$, one has explicit expressions for the two characteristic roots ℓ_1, ℓ_2 of TT' in terms of the elements of TT' in terms of the elements of T. It can easily be shown that ℓ_1, ℓ_2, given (λ_1, λ_2), are conditionally PQD. Furthermore, ℓ_i is st. ↑ in (λ_1, λ_2), for $i = 1, 2$, and (λ_1, λ_2) is associated. Appealing to Th. 3.1.1, we obtain the following:

5.1.1. Result. Let (ℓ_1, ℓ_2) be constructed as described just above. Then (ℓ_1, ℓ_2) is PQD, i.e.

$$P(\ell_1 \leq u_1, \ell_2 \leq u_2) \geq P(\ell_1 \leq u_1) \cdot P(\ell_2 \leq u_2)$$

for all positive real numbers u_1, u_2.

5.2. Characterization of Independence in 2 × 2 Contingency Tables.

In a 2 × 2 contingency table, the experimenter may examine the available data as to the occurrence or nonoccurence of the events $[x_1 \leq a_2]$, where X_1, X_2 are random variables and a_1, a_2 are fixed real numbers.

Jogdeo (1968) determines a suitable family of bivariate distributions such that the independence of the events above is equivalent to the independence of the paired random variables, and introduces a multivariate analog of the bivariate case.

Theorem (Jogdeo, 1968). Let (X_1, X_2, X_3) be POD. Then $X_1, X_2, X_3)$ be POD. Then X_1, X_2, X_3 are independent if and only if

(1) $EX_iX_j = EX_iEX_j$, $i \neq j$, $j = 1, 2, 3$, and

(2) Any one pair, say (X_1, X_2), satisfies $E(X_1X_2|X_3) = E(X_1|X_3)E(X_2|X_3)$.

We extend the above result to the case when the data are observed in a random environment; i.e., when X_1, X_2, and X_3 are only conditionally POD. An immediate consequence of Th. 3.1.1 and Jogdeo's result is the following:

5.2.1. Result. Let X_1, X_2, and X_3, given λ, be conditionally POD. Then X_1, X_2, and X_3 are independent if and only if

(1) $E(X_iX_j|\lambda) = E[X_i|\lambda]E[X_j|\lambda]$, $i \neq j$; $i, j = 1, 2, 3$, and

(2) One of the pairs, say (X_1, X_2), satisfies

$$E[X_1X_2|\lambda, X_3] = E[X_1|\lambda, X_3]E[X_2|\lambda, X_3].$$

5.3. Monotonicity In Conditional Distributions.

A number of inequalities have been derived recently for various multivariate distributions and then used to obtain conservative simultaneous confidence or prediction regions. See, for example, Broemeling (1969), Folks and Antle (1967), Jensen (1969, 1970), Khatri (1967), Shaked (1975), Šidák (1973), Jogdeo (1977, 1978), Tong (1970, 1977), and Ahmed, León, and Proschan (1978). A simple unified method of obtaining such inequalities is by showing that the random variables treated are RTIS.

In this subsection we show that RTIS exists among the random variables, and among certain functions of them that are governed by certain additional multivariate distributions. We rely on Th. 4.6 to demonstrate this RTIS.

(i) The Multivariate Negative Binomial (Negative Multinomial).

This distribution may be generated as follows. Let $X_1, \ldots, X_n$, given λ, be independent random variables with common Poisson density

$$f(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!},\ x = 0, 1, \ldots;\ \lambda > 0,$$

where Λ has the gamma density

$$g(\lambda) = \frac{1}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda},\ \lambda > 0, \alpha > 0.$$

By Th. 4.6, we immediately obtain:

5.3.1. Result. Let $\underline{X}$ be constructed as described just above. Then $\underline{X}$ is RTIS.

(ii) The Pólya Urn Scheme Random Variables.

Suppose that repeated drawings are made from an urn which contains red and black balls, say. Suppose that after each drawing, the ball is replaced, along with $c > 0$ balls of the same color. Further, assume that there are $r > 0$ red balls and $b > 0$ black balls in the urn at the time of the first drawing.

Let $$X_i = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ drawing is red} \\ 0 & \text{if the } i^{\text{th}} \text{ drawing is black,} \end{cases}$$

$i = 1, \ldots, n$. It can be shown that the random variables $X_1, \ldots, X_n$ are stochastically equivalent to a mixture of independent Bernoulli(λ) random variables with $\Lambda \sim \text{Gamma}(\frac{r}{c}, \frac{b}{c})$. By Th. 4.6, we immediately obtain:

5.3.2. Result. Let $\underline{X}$ be constructed as just above. Then $\underline{X}$ is RTIS.

(iii) Positively Dependent by Mixture (PDM) Random Variables.

An important positive dependence property can be defined through the mixture of distribution functions. Let $F_{\underline{X}}(\underline{x})$ denote the cdf of $\underline{X} = (X_1, \ldots, X_n)$. F is called a <u>mixture of distributions</u> if there are distribution functions $G_\lambda^{(i)}(x_i)$(which depend on λ and i) and $H(\lambda)$ such that F admits the representation

$$F_{\underline{X}}(x_1, \ldots, x_n) = \int_\Lambda \prod_{i=1}^n G_\lambda^{(i)}(x_i) dH(\lambda).$$

In particular, if $G_\lambda^i(x_i)$ does not depend on i, i.e., $X_1, \ldots, X_n$, given λ, are conditionally i.i.d, then the unconditional random variables $X_1, \ldots, X_n$ will be PDM. An excellent exposition of this subject, including related interesting results in probability theory and statistics, appears in Shaked (1977).

Now suppose $G_\lambda^i(x_i) = G_\lambda(x)$, where G possesses a TP_2 density in (λ, x). By Th. 4.6, we immediately obtain:

<u>5.3.3. Result.</u> Let $\underline{X}$ be PDM such that $X_1|\lambda$ has a TP_2 density. Then $\underline{X}$ is RTIS.

<u>5.3.4. Remark.</u> The result is of particular interest in reliability theory, for if $X_1, \ldots, X_n$ are the random life lengths of n identical components of a complex system which operates in a random environment, and if, given the environment in which the system operates, $X_1, \ldots, X_n$, given p, are i.i.d. random variables, where p is a measure of the severity of the environment. It is common practice to compute the system life distribution under the assumption that the component life lengths are independent. However, if X_1, given p, has a TP_2 density in (x, p) (e.g.,

$X_1 | p \sim$ Gamma (α, β), then the component life lengths are RTIS Hence, it becomes possible to determine whether under-estimates or over-estimates result from the assumption of component independence, when in fact the component life lengths are RTIS.

5.4. An Extension to Lehmann's Result Concerning A Class of Positive Dependence Measures.

An important class of distributions with positive quadrant dependence is furnished by Theorem 5.4.1 below. This class is essentially an extension of a similar class considered by Lehman (1966, Th. 1), namely, the class containing the measures of association, Kendall's τ, Spearman's ρ, and the quadrant measure q considered by Blomoqvist (1950), since we allow the pairs of random variables to be conditionally PQD (e.g., dependent on a random environment), i.e., $F_i(x_i, y_i) = \int_\Omega F_i^\omega(x_i, y_i) dG(\omega)$, where G is a probability measure defined on $\Omega \subset R$.

Before stating this theorem, we find it convenient to introduce the following definition. We say that real valued functions f and g of n arguments are concordant if, considered as functions of the $i^{\underline{th}}$ coordinate (with all other coordinates held fixed), they are monotone in the same direction; i.e., either both increasing or both decreasing, i = 1, ..., n.

5.4.1. Theorem. Let $(X_1, Y_1)|\omega, \ldots, (X_n, Y_n)|\omega$ be independent pairs of random variables satisfying the conditions of Corollary 3.2.4, with joint distributions $F_1^\omega, \ldots, F_n^\omega$, respectively. Let $f, g : R^n \to R$ be concordant functions. Finally, let $X = f(X_1, \ldots, X_n)$ and $Y = g(Y_1, \ldots, Y_n)$. Then (X, Y) are PQD.

Proof. Using Th. 3.1.1, we may prove Th. 5.4.1 by arguments similar to those used in the proof of Theorem 1 of Lehmann (1966). ||

Consider

$$\tau = \mathrm{Cov}[\mathrm{sgn}(X_2 - X_1), \mathrm{sgn}(Y_2 - Y_1]$$

and

$$\rho_s = 3\mathrm{Cov}[\mathrm{sgn}(X_2 - X_1), \mathrm{sgn}(Y_3 - Y_1)],$$

where the X's and Y's are as described in Th. 5.4.1 just above. τ and ρ are known as Kendall's and Spearman's measures of association, respectively.

Let μ and ν denote the median of the marginal distributions of X and Y, and let f(X) and g(Y) denote the indicator functions of the events $(x > \mu)$ and $Y > \mu)$ respectively. Then

$$q = E[fg + (1 - f)(1 - g) - f(1 - g) - g(1 - f)]$$

is known as Blomqvist's measure of association.

We note here that Lehmann (1966) has shown that if (X_i, Y_i) is PQD, $i = 1, 2, \ldots, n$, then τ, ρ_s, and q are all nonnegative. We may now extend Lehmann's conclusion to:

5.4.2. Result. Under the hypothesis of Th. 5.4.1, the measures of positive dependence of X and Y, Kendell's τ, Spearman's ρ_s, and Blomqvist's q, are all nonnegative.

5.5. Competing Risks with Proportional Failure Rates.

The results of this subsection apply to two models, the competing risk model in biometry, and the series system in reliability.

(1) Competing Risk Model. An organism is subject to k causes of death. If cause i alone were operating, the random lifelength of the or-

ganism would be T_i, $i = 1, \ldots, k$. From data obtained <u>when all causes are operating</u>, the problem is to estimate the joint survival function of $T_1, \ldots, T_k$, the marginal distributions of the $T_1, \ldots, T_k$, as well as various other distributions of interest. See Gail (1975), Sethuraman (1956), Benjamin and Haycocks (1970), Moeschberger and David (1971), and David (1976).

<u>(2) Series System Model.</u> The i^{th} component of a series system has random lifelength T_i, $i = 1, \ldots, k$. From data obtained on system lifelength $T = \min(T_1, \ldots, T_k)$ and the component causing system failure, the problem is to estimate the joint survival function of $T_1, \ldots, T_k$, the (marginal) distributions of the k components, and the distribution of system lifelength.

In the literature, it is often assumed that the random variables $T_1, \ldots, T_k$ are mutually independent. In this subsection we derive bounds on the joint survival function of $T_1, \ldots, T_k$ when the assumption of independence is invalid. (See Elandt-Johnson, 1976, for a discussion of the biometric case and Cohen, 1968, and Langberg, Proschan, and Quinzi, 1977, for discussion of the reliability case). To derive these bounds, we use the theoretical results established in Sec. 4.

For simplicity we use the language of the reliability model; the results, however, apply equally to the biometric model. Let $F_j(t) = P[T_j \leq t]$, $\bar{F}_j(t) = P[T_j > t]$, $F_{\underline{T}}(t_1, \ldots, t_k) = P[T_1 \leq t_1, \ldots, T_k \leq t_k]$, and $\bar{F}_j(t_1, \ldots, t_k) = P[T_1 > t_1, \ldots, T_k > t_k]$. Let

$$r_j(t) = -\frac{\partial}{\partial t_j} \log \bar{F}_{\underline{T}}(t_1, \ldots, t_k)|_{t_1=\ldots=t_k=t},$$

the failure rate of component j at time t given the system is still functioning. Since the system is a series system, then the corresponding system failure rate r(t) is given by

$$r(t) = \sum_{j=1}^{k} r_j(t).$$

In some practical situations it is reasonable to assume proportional failure rate functions which depend on a random environment (Harris and Singpurwalla, 1968). More specifically,

$$r_j(t|\lambda) = a_j r(t|\lambda),$$

where each $a_j > 0$ and $\sum_{j=1}^{k} a_j = 1$, and λ is the random parameter representing the severity of the environment. By Th. 4.11 we immediately obtain:

<u>5.5.1. Result.</u> Let $T_1, \ldots, T_k$ be the lifelengths of the components of a series system with failure rate functions as described just above. Then (a) $T_1, \ldots, T_k$ are RTIS, and (b) $P[T_j > t_j, j = 1, \ldots, k] \geq \prod_{j=1}^{k} P[T_j > t_j]$.

<u>5.5.2. Remark.</u> Theorems 4.4 and 4.11 may also be used to show that random variables $X_1, \ldots, X_n$ having the property that $\min_{1\leq j\leq n} \{a_i X_i\}$ has a Weibull distribution for every choice $a_1 > 0, \ldots, a_n > 0$ are RTIS; i.e.,

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$$P[\min_{1\le i\le n} \{a_i X_i\} > t] = \exp[-k(a_1, \ldots, a_n)t^{\alpha}]$$

for all $t \ge 0$ and some $\alpha > 0$ and $k(a_1, \ldots, a_n) > 0$. (See Lee, 1977, for a detailed analysis of this class of random variables.) Note that the multivariate exponential distribution of Marshall and Olkin, 1967, governs random variables in this class.

5.6. Probability Inequalities Using POD.

In this subsection, by using the POD property, we show how to obtain a number of probability inequalities, for random variables governed by well known distributions. We rely on the theory developed in Section 3 to obtain these inequalities.

(i) - The Absolute Normal with Random Means.

Let (X_1, X_2) have the bivariate normal distribution

$$N\left[(\mu_1, \mu_2), \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right],$$

where $0 < \rho \le \mu_1/\mu_2 < 1/\rho$ or $1/\rho \le \mu_1/\mu_2 \le \rho < 0$. Then for $x_1 > 0$, $x_2 > 0$, Das Gupta et al. (1971) have shown that $(|X_1|, |X_2|)$ is PQD. In fact, this result is the first of its kind for the case of nonzero means.

An interesting extension of the above result, based on Th. 3.1.1, is the following.

5.6.1. Result. Let (X_1, X_2) have the bivariate normal distribution

$$N\left[(\theta, a\theta), \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right]$$

where $0 < \rho \le a < 1/\rho$ or $1/\rho \le a \le \rho < 0$ and θ is a r.v. Then for $x_1 > 0$, $x_2 > 0$, $P[|X_1| \ge x_1, |X_2| \ge x_2] \ge P[|X_1| \ge x_1]P[|X_2| \ge x_2]$.

(ii) Multivariate Noncentral t Distribution.

Let $Y_j = (U_j + \lambda_j)(S_j/\sqrt{\nu_j})^{-1}$, $j = 1, \ldots, m$, where the U's and the S's are all mutually independent, each U_j has a unit normal distribution, and S_j has a χ distribution with ν_j degrees of freedom. Then the joint distribution of $Y_1, \ldots, Y_m$ is a noncentral multivariate t distribution.

There are two ways in which positive dependence can be introduced among the random variables:

(a) Replacing S_j by the same S and ν_j by ν, with S distributed as χ with ν degrees of freedom, so that

$$Y_j = (U_j + \lambda_j)(S/\sqrt{\nu})^{-1}, \; j = 1, \ldots, m.$$

(b) Retaining different and independent S_j's, but requiring $(U_1, \ldots, U_m)$ to have an equi-correlated joint standardized multinormal distribution with a positive correlation coefficient.

In each case, the quantities added to the U's in the numerators are called noncentrality parameters. If every noncentrality parameter is zero the distribution is called a central multivariate t distribution;

An immediate consequence of Th. 3.2.1 is the following:

3.6.2. Result. (1) Each of the multivariate noncentral t random vectors described just above in (a) and (b) is POD; i.e.,

$$P[\bigcap_{j=1}^{m} (Y_j > y_j)] \geq \prod_{\ell} P[y_k > y_k, k \in I_\ell],$$

where the I_ℓ,s are nonempty subsets of $\{1, 2, \ldots, m\}$ whose union is $\{1, 2, \ldots, m\}$.

(2) The same conclusion is true if in (1), $\lambda_j = \lambda$, where λ is a random variable.

(iii) - Multivariate Gamma Distributions.

A distribution with a marginal χ^2 distribution arises naturally in the following way. Consider a random sample of size n determined by n independent vectors $(x_{i1}, \ldots, X_{im})$, $i = 1, \ldots, n$, each vector having the same multinormal distribution with variance-covariance matrix V with each diagonal element equal to 1. Then the statistics $S_j = \sum_{i=1}^{n} (X_{ij} - \bar{X}_{.j})^2$, $j = 1, \ldots, m$, each have a χ^2_{n-1} distribution. The joint distribution of $S_1, \ldots, S_m$ may, following Krishnaiah (1963), be called a multivariate chi-square distribution. It is also called a generalised Rayleigh distribution (see Miller, 1958). For $m = 2$, the conditional distribution of $(X_{11}, \ldots, X_{n1})$ given $(X_{12}, \ldots, X_{n2})$ is that of n independent normal variables with expected values $(X_{12}, \ldots, X_{n2})$ and common variance $(1 - \rho^2)$, where $\rho = \mathrm{Corr}(X_{11}, X_{12})$.

By Th. 3.1.1, we immediately obtain:

5.6.3 Result. Let S_1, S_2 be constructed as just above. Then (S_1, S_2) is PQD.

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UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1. REPORT NUMBER FSU No. M486 AFOSR No. 78-6	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE Two Concepts of Positive Dependence, with Applications in Multivariate Analysis	5. TYPE OF REPORT & PERIOD COVERED Technical Repart 6. PERFORMING ORG. REPORT NUMBER FSU Statistics Report M486
7. AUTHOR(s) Abdul-Hadi N. Ahmed Naftali A. Langberg Ramón V. León Frank Proschan	8. CONTRACT OR GRANT NUMBER(s) AFOSR 78-3678
9. PERFORMING ORGANIZATION NAME & ADDRESS The Florida State University Department of Statistics Tallahassee, Florida 32306	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME & ADDRESS The U.S. Air Force Air Force Office of Scientific Research Bolling Air Force Base, D.C. 20332	12. REPORT DATE November, 1978 13. NUMBER OF PAGES 36
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS

Positive dependence, positive quadrant dependence, right tail increasing, multivariate analysis, conditionally increasing in sequence, inequalities, monotonicity, random parameters, mixtures of distributions, MANOVA, contingency tables, competing risks, reliability, measures of association.

20. ABSTRACT

Two concepts of Positive Dependence, with Applications in Multivariate Analysis. We develop properties and theory for positive orthant dependence, a multivariate extension of Lehmann's positive quadrand dependence, and right tail increasing in sequence dependence, a multivariate extension of Esary and Proschan's bivariate right tail increasing dependence. Applications are then obtained in the form of inequalities and monotonicity in a wide variety of multivariate statistical problems, including MANOVA, contingency tables, dependence measurement, competing risk models, reliability of series systems, and distribution theory.